The scientific objective is to identify the hypotheses that have (or have not) been ruled out by the trial’s results.

Let $U(\theta_{ref}|\hat{\theta}_{obs})$ represent a statistical measure of the consistency between the trial’s result $\hat{\theta} = \hat{\theta}_{obs}$ and the hypothesis $\theta = \theta_{ref}$.

By usual frequentist criteria this measure is equal to the smaller of:

$$P(\hat{\theta} \geq \hat{\theta}_{obs}|\theta = \theta_{ref})$$

$$P(\hat{\theta} \leq \hat{\theta}_{obs}|\theta = \theta_{ref})$$

Reject the hypothesis $\theta = \theta_{ref}$ when $U(\theta_{ref}|\hat{\theta}_{obs})$ is small; specifically when:

$$U(\theta_{ref}|\hat{\theta}_{obs}) < \frac{\alpha}{2}$$
Confidence interval as the non-rejection region

We seek the values of $\theta$ that cannot be rejected; specifically:

- Find the set of $\theta_{ref}$ such that $U(\theta_{ref}|\hat{\theta}_{obs}) \geq \alpha/2$ using $\alpha = 0.05$.
- If $\hat{\theta} \sim \mathcal{N}(\theta, V)$ then the non-rejection region is given by $[\theta_L, \theta_U]$ where

$$
\theta_L = \hat{\theta}_{obs} - 1.96\sqrt{V} \\
\theta_U = \hat{\theta}_{obs} + 1.96\sqrt{V}
$$

- Examples:
  - (A) Treatment effects are only on the mean
  - (B) Treatment effects both the mean and the variance
Example A: treatment affects mean

\[
\begin{align*}
U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) &= 0.00187 \\
\theta_{\text{ref}} &= -0.2 \\
\hat{\theta}_{\text{obs}} &= 0.38
\end{align*}
\]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.00256 \]

\[ \theta_{\text{ref}} = -0.18 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{ref} | \hat{\theta} = \hat{\theta}_{obs}) = 0.00347 \]

\[ \theta_{ref} = -0.16 \]

\[ \hat{\theta}_{obs} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.00466 \]

\[ \theta_{\text{ref}} = -0.14 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

Sampling distribution

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.00621 \]

\[ \theta_{\text{ref}} = -0.12 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{ref} \mid \hat{\theta} = \hat{\theta}_{obs}) = 0.0082 \]
\[ \theta_{ref} = -0.1 \]
\[ \hat{\theta}_{obs} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.01072 \]
\[ \theta_{\text{ref}} = -0.08 \]
\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.0139 \]
\[ \theta_{\text{ref}} = -0.06 \]
\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[
U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.01786
\]

\[
\theta_{\text{ref}} = -0.04
\]

\[
\hat{\theta}_{\text{obs}} = 0.38
\]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.02275 \]

\[ \theta_{\text{ref}} = -0.02 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

$$U(\theta = \theta_{ref} \mid \hat{\theta} = \hat{\theta}_{obs}) = 0.02872$$

$$\theta_{ref} = 0$$

$$\hat{\theta}_{obs} = 0.38$$
Example A: treatment affects mean

$$U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.03593$$

$$\theta_{\text{ref}} = 0.02$$

$$\hat{\theta}_{\text{obs}} = 0.38$$
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.04457 \]
\[ \theta_{\text{ref}} = 0.04 \]
\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.0548 \]

\[ \theta_{\text{ref}} = 0.06 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.06681 \]
\[ \theta_{\text{ref}} = 0.08 \]
\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.08076 \]
\[ \theta_{\text{ref}} = 0.1 \]
\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

$\theta_{obs} = 0.38$
$\theta_{ref} = 0.12$
$U(\theta = \theta_{ref} | \hat{\theta} = \hat{\theta}_{obs}) = 0.0968$
Example A: treatment affects mean

Sampling distribution

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.11507 \]
\[ \theta_{\text{ref}} = 0.14 \]
\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{ref} \mid \hat{\theta} = \hat{\theta}_{obs}) = 0.13567 \]

\[ \theta_{ref} = 0.16 \]

\[ \hat{\theta}_{obs} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.15866 \]
\[ \theta_{\text{ref}} = 0.18 \]
\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

U(θ = θ_{ref} | \hat{θ} = \hat{θ}_{obs}) = 0.18406
θ_{ref} = 0.2
\hat{θ}_{obs} = 0.38
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.21186 \]

\[ \theta_{\text{ref}} = 0.22 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[ U(θ = θ_{\text{ref}} \mid \hat{θ} = \hat{θ}_{\text{obs}}) = 0.24196 \]

\[ θ_{\text{ref}} = 0.24 \]

\[ \hat{θ}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[
U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.27425
\]

\[
\theta_{\text{ref}} = 0.26
\]

\[
\hat{\theta}_{\text{obs}} = 0.38
\]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.30854 \]

\[ \theta_{\text{ref}} = 0.28 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[
U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.34458
\]

\[
\theta_{\text{ref}} = 0.3
\]

\[
\hat{\theta}_{\text{obs}} = 0.38
\]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.38209 \]
\[ \theta_{\text{ref}} = 0.32 \]
\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.42074 \]

\[ \theta_{\text{ref}} = 0.34 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.46017 \]

\[ \theta_{\text{ref}} = 0.36 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{ref} \mid \hat{\theta} = \hat{\theta}_{obs}) = 0.5 \]

\[ \theta_{ref} = 0.38 \]

\[ \hat{\theta}_{obs} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.46017 \]

\[ \theta_{\text{ref}} = 0.4 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

$$U(\theta = \theta_{ref} | \hat{\theta} = \hat{\theta}_{obs}) = 0.42074$$

$$\theta_{ref} = 0.42$$

$$\hat{\theta}_{obs} = 0.38$$
Example A: treatment affects mean

\[ \text{Sampling distribution} \]

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.38209 \]

\[ \theta_{\text{ref}} = 0.44 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.34458 \]

\[ \theta_{\text{ref}} = 0.46 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.30854 \]
\[ \theta_{\text{ref}} = 0.48 \]
\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

$$\text{Sampling distribution}$$

$$0.5 \theta \theta^\text{obs} \sim U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.27425$$

$$\theta_{\text{ref}} = 0.5$$

$$\hat{\theta}_{\text{obs}} = 0.38$$
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.24196 \]

\[ \theta_{\text{ref}} = 0.52 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

$$U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.21186$$

$$\theta_{\text{ref}} = 0.54$$

$$\hat{\theta}_{\text{obs}} = 0.38$$

Sampling distribution
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.18406 \]
\[ \theta_{\text{ref}} = 0.56 \]
\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.15866 \]

\[ \theta_{\text{ref}} = 0.58 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

Sampling distribution

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.13567 \]
\[ \theta_{\text{ref}} = 0.6 \]
\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

Sampling distribution

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.11507 \]

\( \theta_{\text{ref}} = 0.62 \)

\( \hat{\theta}_{\text{obs}} = 0.38 \)
Example A: treatment affects mean

Setting and Objectives

Confidence Interval Interpretation

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.0968 \]
\[ \theta_{\text{ref}} = 0.64 \]
\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[
U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.08076
\]

\[
\theta_{\text{ref}} = 0.66
\]

\[
\hat{\theta}_{\text{obs}} = 0.38
\]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.06681 \]

\[ \theta_{\text{ref}} = 0.68 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.0548 \]
\[ \theta_{\text{ref}} = 0.7 \]
\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.04457 \]

\[ \theta_{\text{ref}} = 0.72 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.03593 \]

\[ \theta_{\text{ref}} = 0.74 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[
\begin{align*}
U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) &= 0.02872 \\
\theta_{\text{ref}} &= 0.76 \\
\hat{\theta}_{\text{obs}} &= 0.38
\end{align*}
\]
Example A: treatment affects mean

\[
U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.02275 \\
\theta_{\text{ref}} = 0.78 \\
\hat{\theta}_{\text{obs}} = 0.38
\]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.01786 \]

\[ \theta_{\text{ref}} = 0.8 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.0139 \]

\[ \theta_{\text{ref}} = 0.82 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

Sampling distribution

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.01072 \]

\[ \theta_{\text{ref}} = 0.84 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

$$U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.0082$$

$$\theta_{\text{ref}} = 0.86$$

$$\theta_{\text{obs}} = 0.38$$
Example A: treatment affects mean

Sampling distribution

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.00621 \]

\[ \theta_{\text{ref}} = 0.88 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[
U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.00466
\]

\[
\theta_{\text{ref}} = 0.9
\]

\[
\hat{\theta}_{\text{obs}} = 0.38
\]
Example A: treatment affects mean

Sampling distribution

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.00347 \]

\( \theta_{\text{ref}} = 0.92 \)

\( \hat{\theta}_{\text{obs}} = 0.38 \)
Example A: treatment affects mean

Sampling distribution

\( U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.00256 \)

\( \theta_{\text{ref}} = 0.94 \)

\( \hat{\theta}_{\text{obs}} = 0.38 \)
Example A: treatment affects mean

Confidence Interval Interpretation

U(θ = θ_{ref} \mid \hat{θ} = \hat{θ}_{obs}) = 0.00187

θ_{ref} = 0.96

\hat{θ}_{obs} = 0.38
Example A: treatment affects mean

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.00135 \]

\[ \theta_{\text{ref}} = 0.98 \]

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: treatment affects mean

\[ U(\theta = \theta_{ref} \mid \hat{\theta} = \hat{\theta}_{obs}) = 0.00097 \]
\[ \theta_{ref} = 1 \]
\[ \hat{\theta}_{obs} = 0.38 \]
Example A: Non-rejection interval

Sampling distribution

$\hat{\theta}_{\text{obs}} = 0.38$
Example A: Non-rejection interval

\[ \hat{\theta}_{\text{obs}} = 0.38 \]

\[ \theta_L \quad \hat{\theta} \]

Sampling distribution
Example A: Non-rejection interval

\[ \hat{\theta}_{\text{obs}} = 0.38 \]
Example A: Non-rejection interval

\[ \hat{\theta}_{\text{obs}} = 0.38 \]

CI (non-reject) = (-0.01, 0.77)
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 4e^{-05} \]

\[ \theta_{\text{ref}} = 0.81873 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ \text{Sampling distribution} \]

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 9 \times 10^{-5} \]

\[ \theta_{\text{ref}} = 0.83527 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ \theta \sim U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.00017 \]

\[ \theta_{\text{ref}} = 0.85214 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.00032 \]

\[ \theta_{\text{ref}} = 0.86936 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.00059 \]

\[ \theta_{\text{ref}} = 0.88692 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.00103 \]

\[ \theta_{\text{ref}} = 0.90484 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.00286 \]
\[ \theta_{\text{ref}} = 0.94176 \]
\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

$$U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.00453$$

$$\theta_{\text{ref}} = 0.96079$$

$$\hat{\theta}_{\text{obs}} = 1.46228$$
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.00696 \]

\[ \theta_{\text{ref}} = 0.9802 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.0104 \]

\[ \theta_{\text{ref}} = 1 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{ref} | \hat{\theta} = \hat{\theta}_{obs}) = 0.01513 \]
\[ \theta_{ref} = 1.0202 \]
\[ \hat{\theta}_{obs} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.02145 \]

\[ \theta_{\text{ref}} = 1.04081 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.02967 \]
\[ \theta_{\text{ref}} = 1.06184 \]
\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

The sampling distribution

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.04012 \]

\[ \theta_{\text{ref}} = 1.08329 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{ref} | \hat{\theta} = \hat{\theta}_{obs}) = 0.05309 \]
\[ \theta_{ref} = 1.10517 \]
\[ \hat{\theta}_{obs} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.06882 \]
\[ \theta_{\text{ref}} = 1.1275 \]
\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.08751 \]
\[ \theta_{\text{ref}} = 1.15027 \]
\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[
U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.10928
\]

\[
\theta_{\text{ref}} = 1.17351
\]

\[
\hat{\theta}_{\text{obs}} = 1.46228
\]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \theta_{\text{obs}}) = 0.13414 \]

\[ \theta_{\text{ref}} = 1.19722 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

$$U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.16205$$

$$\theta_{\text{ref}} = 1.2214$$

$$\hat{\theta}_{\text{obs}} = 1.46228$$
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{ref} | \hat{\theta} = \hat{\theta}_{obs}) = 0.19282 \]

\[ \theta_{ref} = 1.24608 \]

\[ \hat{\theta}_{obs} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.22622 \]

\[ \theta_{\text{ref}} = 1.27125 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ \text{Sampling distribution} \]

\[
U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.2619
\]

\[
\theta_{\text{ref}} = 1.29693
\]

\[
\hat{\theta}_{\text{obs}} = 1.46228
\]
Example B: treatment affects mean and variance

\begin{align*}
U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) &= 0.29949 \\
\theta_{\text{ref}} &= 1.32313 \\
\hat{\theta}_{\text{obs}} &= 1.46228
\end{align*}
Example B: treatment affects mean and variance

\[
U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.33855 \\
\theta_{\text{ref}} = 1.34986 \\
\hat{\theta}_{\text{obs}} = 1.46228
\]

1.35 \ \theta \ \theta^\circ \ U( \theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.33855 \\
\theta_{\text{ref}} = 1.34986 \\
\hat{\theta}_{\text{obs}} = 1.46228
Example B: treatment affects mean and variance

\begin{equation}
U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.37859
\end{equation}

\begin{align*}
\theta_{\text{ref}} &= 1.37713 \\
\hat{\theta}_{\text{obs}} &= 1.46228
\end{align*}
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.41916 \]
\[ \theta_{\text{ref}} = 1.40495 \]
\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.45977 \]
\[ \theta_{\text{ref}} = 1.43333 \]
\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.5 \]

\[ \theta_{\text{ref}} = 1.46228 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[
U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.46057
\]

\[
\theta_{\text{ref}} = 1.49182
\]

\[
\hat{\theta}_{\text{obs}} = 1.46228
\]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.42228 \]
\[ \theta_{\text{ref}} = 1.52196 \]
\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.38546 \]
\[ \theta_{\text{ref}} = 1.55271 \]
\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{ref} | \hat{\theta} = \hat{\theta}_{obs}) = 0.35033 \]
\[ \theta_{ref} = 1.58407 \]
\[ \hat{\theta}_{obs} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.3171 \]

\[ \theta_{\text{ref}} = 1.61607 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.2859 \]
\[ \theta_{\text{ref}} = 1.64872 \]
\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.25681 \]
\[ \theta_{\text{ref}} = 1.68203 \]
\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

Sampling distribution

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.22987 \]

\[ \theta_{\text{ref}} = 1.71601 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.20507 \]

\[ \theta_{\text{ref}} = 1.75067 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[
U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.18238
\]
\[
\theta_{\text{ref}} = 1.78604
\]
\[
\hat{\theta}_{\text{obs}} = 1.46228
\]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.16172 \]
\[ \theta_{\text{ref}} = 1.82212 \]
\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.14302 \]

\[
\begin{align*}
\theta_{\text{ref}} &= 1.85893 \\
\hat{\theta}_{\text{obs}} &= 1.46228
\end{align*}
\]
Example B: treatment affects mean and variance

\[ \mathbf{U}(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.12616 \]

\[ \theta_{\text{ref}} = 1.89648 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.11103 \]
\[ \theta_{\text{ref}} = 1.93479 \]
\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.0975 \]

\[ \theta_{\text{ref}} = 1.97388 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[
U(\theta = \theta_{ref} \mid \hat{\theta} = \hat{\theta}_{obs}) = 0.08546
\]

\[
\theta_{ref} = 2.01375
\]

\[
\hat{\theta}_{obs} = 1.46228
\]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} | \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.07477 \]
\[ \theta_{\text{ref}} = 2.05443 \]
\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.06532 \]
\[ \theta_{\text{ref}} = 2.09594 \]
\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[
\begin{align*}
U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) &= 0.05697 \\
\theta_{\text{ref}} &= 2.13828 \\
\hat{\theta}_{\text{obs}} &= 1.46228
\end{align*}
\]
Example B: treatment affects mean and variance

\[
U(\theta = \theta_{ref} | \hat{\theta} = \hat{\theta}_{obs}) = 0.04964
\]

\[
\theta_{ref} = 2.18147
\]

\[
\hat{\theta}_{obs} = 1.46228
\]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.04319 \]
\[ \theta_{\text{ref}} = 2.22554 \]
\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{ref} \mid \hat{\theta} = \hat{\theta}_{obs}) = 0.03755 \]

\[ \theta_{ref} = 2.2705 \]

\[ \hat{\theta}_{obs} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.03262 \]
\[ \theta_{\text{ref}} = 2.31637 \]
\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.02832 \]

\[ \theta_{\text{ref}} = 2.36316 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.02457 \]

\[ \theta_{\text{ref}} = 2.4109 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.02131 \]
\[ \theta_{\text{ref}} = 2.4596 \]
\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.01848 \]

\[ \theta_{\text{ref}} = 2.50929 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.01602 \]
\[ \theta_{\text{ref}} = 2.55998 \]
\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.01389 \]

\[ \theta_{\text{ref}} = 2.6117 \]

\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

Sampling distribution

\[ U(\theta = \theta_{\text{ref}} \mid \hat{\theta} = \hat{\theta}_{\text{obs}}) = 0.01204 \]
\[ \theta_{\text{ref}} = 2.66446 \]
\[ \hat{\theta}_{\text{obs}} = 1.46228 \]
Example B: treatment affects mean and variance

$$U(\theta = \theta_{ref} \mid \hat{\theta} = \hat{\theta}_{obs}) = 0.01044$$

$$\theta_{ref} = 2.71828$$

$$\hat{\theta}_{obs} = 1.46228$$
Example B: Non-rejection interval

\[ \hat{\theta}_{\text{obs}} = 1.46 \]
Example B: Non-rejection interval

\[ \theta_\text{obs} = 1.46 \]
Example B: Non-rejection interval

\[ \hat{\theta}_{\text{obs}} = 1.46 \]

\[ \theta, \theta_U \]

Sampling distribution
Example B: Non-rejection interval

\[ \hat{\theta}_{\text{obs}} = 1.46 \]

CI (non-reject) = (1.05, 2.41)
Note: confidence intervals as pivotal quantities vs non-rejection regions

There are two ways to construct a confidence interval:

1. As a non-rejection region as defined above.
   - $\theta_L$ satisfies: $P(\hat{\theta} \geq \hat{\theta}_{obs} | \theta = \theta_L) = 0.025$
   - $\theta_U$ satisfies: $P(\hat{\theta} \leq \hat{\theta}_{obs} | \theta = \theta_U) = 0.025$
   - The non-rejection region is defined in parameter space.
Note: confidence intervals as pivotal quantities vs non-rejection regions

2. From “pivotal quantity”:

\[ P \left( -1.96 \leq \frac{\hat{\theta} - \theta}{\sqrt{\hat{V}}} \leq 1.96 \right) = 0.05 \]

where \( \hat{V} = \text{var}(\hat{\theta}) \)

- Pivoting this probability statement gives

\[ P \left( \hat{\theta} - 1.96\sqrt{\hat{V}} \leq \theta \leq \hat{\theta} + 1.96\sqrt{\hat{V}} \right) \]

- Thus:
  \( \hat{\theta}_L = \hat{\theta} - 1.96\sqrt{\hat{V}} \)
  \( \hat{\theta}_U = \hat{\theta} + 1.96\sqrt{\hat{V}} \)

- This confidence interval is defined in *sample space*. 
Note: confidence intervals as pivotal quantities vs non-rejection regions

Notes:
- From a scientific perspective we are interested in statements about parameter space (non-rejection confidence intervals).
- Non-rejection and pivotal confidence intervals will not necessarily agree.
- For non-rejection regions we need to know how the variance changes with the mean. We usually do not know the mean-variance relationship - careful consideration is necessary (to be discussed further).
Example A: Non-rejection vs pivotal CI
Example A: Non-rejection vs pivotal CI

\[ \hat{\theta}_{\text{obs}} = 0.38 \]

\[ \text{CI (pivotal)} = (-0.01, 0.77) \]
Example A: Non-rejection vs pivotal CI

\[ \hat{\theta}_{\text{obs}} = 0.38 \]

CI (pivotal) = \((-0.01, 0.77)\)

CI (non-reject) = \((-0.01, 0.77)\)
Example B: Non-rejection vs pivotal CI

\[ \hat{\theta}_{\text{obs}} = 1.46 \]
Example B: Non-rejection vs pivotal CI

\[ \hat{\theta}_{obs} = 1.46 \]

CI (pivotal) = (0.89, 2.04)
Example B: Non-rejection vs pivotal CI

\[ \hat{\theta}_{\text{obs}} = 1.46 \]

CI (pivotal) = (0.89, 2.04)

CI (non-reject) = (1.05, 2.41)